Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

VOL. III. No. 16. AUGUST 2, 1906.

THE JOURNAL OF PHILOSOPHY
PSYCHOLOGY AND SCIENTIFIC METHODS

THE GROUP CONCEPT IN THE SERVICE OF PHILOSOPHY

THE best service that can be rendered to philosophy by any science is the creation of concepts which can be made tools for the general study of experience. To take over bodily into philosophy any scientific method is a policy whose mistake is attested by failures frequent and glorious enough; but for conceptual engines we have depended, and must depend, upon what the sciences are producing. Of these engines few have been devised of such scope and power as the concept 'group' which the mathematicians have been forging out for a hundred and fifty years.[1] This concept is an achievement of what we may call the morphology of mathematics; and by due refinement, discarding what is specifically mathematical, it can be set free for a career not less extensive than the entire morphology of experience. To effect this release, we have first of all to study the 'group' idea as it exists in mathematics, in an attempt to exhibit its logical essence, an attempt to which this paper is devoted. A succeeding paper will outline the field of its application.

The idea 'group' fills in extraordinary measure three typical and apparently divergent ambitions of the mathematical concept-builder: it is at once highly specialized, highly significant and highly simple. It is highly specialized, that is, limited; it is removed by a great depth of determination from the empty idea of a class of objects in general. We know, on general principles, that the amount of meaning carried by any idea, and extractable from it, bears some relation to its restrictedness; the properties necessary in order that a class of objects should qualify as a group are striking and—if we may speak of ideas thus statistically—rare. But a concept may be ideally restricted and yet empirically hospitable; it is so if what it excludes is *excluded by reality also*. If a concept thus adds to its specialization significance, it is what we mean in mathematics, and elsewhere, by a *powerful* concept. The group concept is powerful in this sense: it concentrates a large range of the actual business of mathematics into itself; and demonstrations made upon it have that quality which is called in the weird language of mathematical esthetics, 'elegance.' But thirdly, it is highly simple; and here we come upon a notion much used and little understood—perhaps because it itself is simple—an ideal which expresses itself usually in the demand that the concept shall be definable in few and obvious terms. In brief, the mathematician, in the union of these antagonistic motives, is trying to get 'near to nature'; and the degree to which he has succeeded sufficiently explains the importance of his product to the morphology of experience. Let me state at once, in rough, the salient character of a group, as a basis from which to reach by stages a precise definition.

---

[1] Bibliography in 'Transactions of American Mathematical Society,' April, 1906; 'Note on Definitions of Abstract Groups, etc.,' by E. V. Huntington.

A group is a class of objects so related to each other that a rule may be stated whereby, when any two members of the class are given, a third member is pointed out, or determined. Whole numbers, for example, are so related to each other that, by a rule of addition, any two of them determine a third. If the class be rational numbers, the rule of multiplication has the required property. For forces, angles, etc., similar rules may be alleged; the constitution of these worlds of objects permits any two of the given class thus to govern, or infallibly indicate, a third.

From the logical point of view, this definition has a very striking peculiarity. A class of objects is usually defined by naming the characters of the typical member of the class; the relations which members of the class bear to each other are not mentioned. The group definition, on the contrary, says nothing whatever about the character of the individual members of the group, and is concerned solely with the relations among them. It is the business of ordinary class definition to concentrate upon, and confine to, the individual all the distinguishing criteria of a class, and to consider peculiar relationships as resultant from this character. The group definition abandons this line of attack upon the object of knowledge: it begins with the relation.

Now the 'rule' which plays so conspicuous a part in the definition is nothing else than this relation transformed and fluent—thought of as *active*. The characteristic relationship of group members is triadic, and the rule exhibits this relationship as clothing itself with its full complement of terms; two terms of a triad being filled with two given members of the class, the relation, under the guise of a process, is conceived to discover its missing term. There is no logical necessity, so far as definition is concerned, for thus mobilizing the group relation and turning it into a rule; this is a mere accident of the mathematical exploitation of the group structure for purposes of calculation. The essential point is, that if the whole numbers, for instance, constitute a group, it is not on their own merits, as members of the class whole numbers, but by virtue primarily of a certain triadic relation into which the various whole numbers can be thrown: that, namely, of *a, b* and *c,* when *a* plus *b* is said to *equal c.*

The fact that the brunt of definition of a group falls upon the group relation or rule does not mean that the individual is lost from sight. Just as the description of the individual, in ordinary class definition, limits the variety of possible relationships within the class; so the specification of relationship limits the range of the type of individual that can stand therein. It is true that of all conceivable classes of objects having a plurality of members, there is perhaps none for which some ingenious rule might not be devised to comply with the rough definition we have given. For so disorderly a class as, let us say, all the ships on the Atlantic, there might conceivably be a convention to the effect that any two ships shall indicate that other ship whose latitude at any time is most nearly their mean latitude. But in proportion as the nature of the rule is brought to precise expression we shall find the variety of individual types that can support that rule rapidly lessened. By excluding the artificial, ingenious and conventional types of rule the mathematician excludes at the same time the more contingent, accidental and limited sorts of classes; there is a genuine *rapprochement* between rule and type of object, until the rule becomes, so to speak, the *natural law* of the group—as addition is, in a genetic sense, the natural law of whole numbers. It is of extreme interest to observe how the mathematician has made precise the statement of the properties of the kind of rule thus vaguely described.

Let us first require of the rule, simply, that it point out always *one and only one* third member, any two being given. The disorderly class above suggested is immediately excluded, and with it a vast army. For whatever the character by which the rule identifies the required object, it must be a character which one of the class *must* have, and no two of the class *can* have. Very few classes can boast, on the one hand, such intrinsic individuation and, on the other, such complete representation of all points in that individual order which the rule might possibly call for. Most classes of objects, in their principle of individuation, are parasitic, making use of the already individual points of some field, such as space, to identify their members: none of these can yield a group rule.

Consider, secondly, an important series of classes—the species and genera of organisms in their genetic relations. Imagine the species buffalo, ideally complete in infinite past time. The rule that any two buffaloes indicate their nearest common male ancestor complies with the requirements so far made. The ancestral series is necessarily without gap, and the individuation is intrinsic to the species. But from the point of view of the group, such a rule has a serious defect: there is a large segment of the class which the rule can never reach in its determinations; the female ancestors are never identified. It is required of the group rule that it be possible to determine by its use *all* members of the class; and this requires of the class that its intrinsic order of individuation involve all its members homogeneously. Let us now state, in its own brief terms, a technical definition of a group. Let there be a well-defined[2] class, or assemblage, of objects, K. Then K will be a group with reference to any statable rule which complies with the following requirements:

I. Given any two members, or elements, a and b, of K, there is determined by the rule a unique object, x, which is also an element of K.

II. Every element of K is determined by the rule from two other objects, either one of which being chosen at will from K, the other is thereby fixed, and is an element of K.[3]

III. Any three elements of the assemblage being given, if the first is paired with the second according to the rule, and their result with the third, the object thus determined is the same object as is found by pairing the second with the third, and then the first with their result. In other words, the law of association holds.[4]

---

[2] Defined, that is, by a condition which every object in the universe either does or does not satisfy, so that there is no ambiguous or doubtful margin of objects.

[3] There is no whole number, for example, which is not a sum of two others, properly chosen; any (other) whole number may be taken as one of a pair of addends to make that sum, in which case the second member of the pair is also a whole number, uniquely fixed.

[4] In terms of the relationship R, these postulates become:

I. Given members a and b, there is a unique object, x, such that $R(a, b, x)$; and x is a member of K.

II. Given members, a and b, there is a unique object, y, such that $R(a, y, b)$; and a unique object, z, such that $R(z, a, b)$; and y and z are members of K.

These two can obviously be put into one, to the effect that:

Given members, a and b, there are unique objects, x, y, z, such that $R(a, b, x)$, $R(a, y, b)$, $R(z, a, b)$; and x, y and z are members of K.

III. If we have $R((a, b), c, v)$ and also $R(a, (b, c), u)$, then v and u are identical (the convention (a, b) meaning 'the element determined by a and b').

In this definition, postulates I. and II. clearly supplement one another. Distinguishing in the triad two functions, that of the determining pair and that of the determined element, postulate I. demands that for every pair there be a determined element; while postulate II. demands that for every element there be a determining pair, or series of determining pairs. Postulates I. and II. do not of themselves, however, instruct us which element in any given triad is to be regarded as the determined element, and which pair the determining pair. Postulate II. intimates that no matter which of the three couples in any triad are given, the remaining element is determined; so that wherever it is possible to state one rule, it is possible to state three rules, which equally comply with the conditions. If a and b determine x, then a and x will determine b by another rule, and b and x will determine a by still another. If whole numbers are a group under addition, they will also be a group under substraction,[5] so far as these two postulates are concerned. Postulates I. and II. are, in fact, equivalent to the following summary definition: A group is an assemblage whose elements sustain triadic relationships of the form R; R being such that two elements are *necessary,* and any two are *sufficient,* in any two terms of the triad, to fix uniquely the third term of the triad.

But the three rules which govern any triad are, in an important sense, phases of one rule. Addition and subtraction, for instance, belong to each other as addition and multiplication do not. And

---

The language of R has the advantages and disadvantages of symmetry. It must not be forgotten that the original trend of determination is from a and b to x, in $R(a, b, x)$, and not to y and z in $R(a, y, b)$ and $R(z, a, b)$; only the first defines the specific rule. There is no postulate to the effect that y and z are discoverable; and if they are so discoverable, the rules by which they are found are genetically 'inverse' to the rule of $R(a, b, x)$. The language of R further obscures certain important ordinal distinctions between postulates I. and II.; distinctions of the sort brought out in the paper of E. V. Huntington (superseded by his own later work, but in this respect not improved on) in *Bulletin of the American Mathematical Society,* 2d Series, April, 1902. It is II. that gives the concept its mathematical stamp; postulate I. is presupposed in II. and is on a different logical plane. The relation between them might be expressed by calling II. the postulate of determined determinants. The language of R is appropriate for expressing results, and for exhibiting the philosophical character of the finished concept, but not for showing the genetic meaning of the group.

[5] The circumstance that the two kinds of subtraction are identical is a concomitant of the circumstance that for addition the order of the two addends is indifferent—if $R(a, b, x)$, then also $R(b, a, x)$; the commutative law holds; the group is a so-called Abelian group. In general, the functions of a and b in the triad are different, so that their order can not be inverted without altering the object determined, in which case the three rules are all different.

there are reasons for regarding *one* of the three rules of a triad as the group rule, par excellence, from which the other two are derivative; one as direct, the others as 'inverse.' These reasons, whatever they may be, are embodied in the third postulate, which addition, multiplication, etc., satisfy, whereas subtraction and division do not.

This third postulate is something of a mystery. It offers no illuminating insight into its essential motives. It seems mechanical, extraneous, and a disturbance to the neat internal harmony of the others. It has often appealed to the mathematician as a conclusion, or theorem, rather than as a first premise. But if it is deducible from something more primary, that something has not yet been formulated. I surmise that this postulate of association *is* a deduction; but that its premises lie in a region outside mathematics proper, so that the mathematician will always have to content himself with the cumbrous law as a first principle. In what follows I shall make an attempt to find its sources. I turn to an analysis of the concept now definitely before us.[6]

The business of mathematics is to make identity do work. The difference between an equation which has a root and an identical equation, so-called, which is analytic, barren and stationary, illustrates the motive in question. If we could discover within these three postulates the essential unity of the group concept, and at the same time the secret of its power, we should find it in some such working identity, some asserted or demanded agreement between things conceptually independent. There is, in fact, a fundamental duality in the group concept, a duality which each one of the postulates in its own way partially expounds and synthesizes.

We have said that the group definition addresses itself to the relation between objects rather than to the individual object. The ideal of this relational type of definition is to ignore the individual, and to describe its class as 'the totality of all possible terms of the relation *R*.' But the group definition does not attain this ideal detachment. There is no such thing, for example, as the totality of all possible terms that can stand in additive relations. Whole numbers, real numbers, numbers and continua of very various kinds, alike constitute groups under addition. There is no unmistakable implication and marriage of rule and class.[7] Hence it is that two data are necessary to designate any actual group. A group is not fully described as 'the group of multiplications,' but as 'the group of multiplications *among* rational numbers,' or, as 'the group of rational numbers *with reference to* multiplication.' A group is, in reality, a double genus: it undertakes to mean at the same moment objects of two distinct orders,—things as well as relations, a class of particulars as well as a kind of class. The working power of our concept is measured by the degree of independence between these two sets of objects which it brings together, and by the degree of unity which it effects. The fundamental deed of the group concept may be thus stated: It defines a correspondence between the relatively independent domains of a dyadic class relation and a triadic relation of the form *R*, above described.

---

[6] Thesis: The primordial data of mathematics must always be pragmatically defined, in the sense that they express *particular results* of the character of the concept, and not the character itself. They follow precisely the form of the old Latin *ut* clauses; the object to be defined is 'of such a nature that as a result' m equals n. Hence it is that definition takes the form of a set of postulates. And hence it is that no set of postulates can be proven 'sufficient' for an assemblage of real objects (*e. g.*, for absolute continuous magnitude as an object of experience), though sufficient connection may very well be made with a type of theory *applicable* to continua, when that theory already exists as a body of fundamental theorems. Weber's distinction here is fundamental. See 'Lehrbuch der Algebra,' II., S. 6, 'Die Eigenschaften der Gruppen. u. s. w.'

The most obvious phase of the correspondence in question is *coincidence in range*. In so far as connotation is considered to determine range, or denotation, the ordinary class relation is virtually made an active rule—is mobilized in a manner analogous to the flux of the group relation. But in contrast with the group process, it reaches wholeness at a bound. It may indeed be conceived to operate serially, in a chain of *recognitions*—in which case it may develop in the form of a 'Kette,' as in the number series; but this chain is, at the statement of the definition, ideally complete; the membership and outline of the class are fixed. If we presuppose that the class rule has completed its work, and the assemblage *K* is completely made up; and if we then allow the group rule in its triple character to run at large within *K*, using every possible pair of elements, in every possible way, to determine new terms: we already know that in the totality of these new terms (1) nothing is added to the assemblage, and (2) nothing of the assemblage is left out.[8] The domain of the group rule thus obviously coincides with the domain of the class rule. But the significance of any agreement is in proportion to the degree of independence between the agreeing processes; and the group rule has a greater independence of K than our present presupposition shows. Complete independence is, of course, impossible; for the group process does not work in the void, as the class process does—it requires a datum of two objects in order to make a beginning; but for certain fundamental groups a *base* of two elements can be so chosen that the group process, by adding its successive results to the base, will run through the whole assemblage, determining every member without omission. And for any group whatever there is a *class* of objects, included within the group, from which as a base the group process will thus generate, or rather, regenerate, the whole assemblage. To measure the full worth of the demanded coincidence of range of these two processes we should have to determine this base. For our purpose it will be enough if we locate it within the group by establishing one necessary element thereof.

---

[7] There is, however, a peculiar and obvious appropriateness between certain rules and certain classes. Addition is more at home among whole numbers than anywhere else. The group concept will not be finally clear until the distinction is made between the original and the derived classes for any rule, and the perfect mutuality of rule and object has been established, for these originals.

[8] By postulates I. and II. the active group relation, conceived as a triple rule determines always class members, and nothing but class members. Thus, a process like multiplication-division, which working at large among whole numbers deposits its fruit sometimes within the class and sometimes without,

There appears to be nothing in the definition of the group process itself which would enable us to single out any region of an assemblage in preference to another as a base. The group idea is emancipated from the peculiarities of the objects both by its primary interest in their relations and by its ideal generality. The group process is impartial in its concern with the several elements of the assemblage; the distribution of the triads into which the assemblage crystallizes under its group rule is perfectly homogeneous. Considering the totality of these triads, it appears from postulates I. and II. that every element is found *with equal frequency* in each of the three positions in the triad, determining and determined; and that all elements occur *with the same frequency*. According to this ideal impartiality, in thinking of the pairs into which any given element can enter, the pair *with itself* is not omitted. For every element m there must be elements x, y, z, such that $R(m, m, x)$, $R(m, y, m)$, $R(z, m, m)$.

But in the moment of attaining this perfection of indifference to the elements the generality of the process seems to break off its own point. By postulate III. it can be shown that y and z in $R(m, y, m)$ and $R(z, m, m)$ are always identical. Also that in $R(m, y, m)$, $R(n, y', n)$, $R(p, y'', p)$, etc., throughout the assemblage, y, y', y'', etc., are all identical, and so also identical with the corresponding z, z', z''. This identical element, which thus con-

---

can not be a group rule for whole numbers. By Postulate II., every member of the class lies within the sphere of determination.

stantly reappears in all cases of self-determination, may well receive a special symbol, i. For it, $R(i, i, i)$ holds; and it is the only element that can occupy all three terms of R at once. Such an element is the *zero* of addition, and the *unity* of multiplication. All the exceptionalness that was set aside by allowing elements to pair with themselves, is concentrated upon this salient member. The significance of the postulate of association we may presume is bound up with its ability to precipitate the element i from the homogeneous body of the assemblage. We have now to come into closer quarters with that postulate.

It is the work of the group process to transmute determinations of two elements into determinations of one. This requires that the determining elements be themselves determinate. Now there are two ways in which a pair of objects may be determinate: either with reference to one another; or, in relative independence of one another, with reference to a common third. If every pair of elements of an assemblage were determinate in the second manner, and if the common third of all pairs were the same object, we should have reason to characterize their determinations as *absolute,* in contrast with the merely relative determinations of the former type—a kind of absolute which, as usual, means simply, relative to a third. From the point of view of the class relation merely, the adoption of any point of reference for 'absolute' values is purely *arbitrary.* The points of a line are in order, the relations of positions and figures in space and of discrete objects submit to mathematical reasoning, without the assumption of any *origin.* But

(α) No *associative* process can be defined without reference of the determining and determined objects to an origin (taking this term in as wide a sense as is necessary to cover the variety of fundamental groups).

(β) No associative process can be defined wherein the relation of the determining objects to each other is a constituent function.[9] On the other hand,

(γ) That any rule be associative, it is sufficient that in the determinations of x, in $R(a, b, x)$, the determinations of a and b, with reference to some origin of absolute values, shall exist unaltered.[10]

---

[9] No rules, that is, can be associative in which x, in $R(a, b, x)$, is a function of $b:a$. K being all numbers, the rules that $x = b - a$; $x - b = b - a$; $x/b = b/a$; $x = (b - a)(b + a)$; etc.; or, K being the points in space, the rule that x be the middle point between b and a, etc.—are for this reason not associative.

[10] As most obviously in addition of numbers; also in addition of displacements referred to Cartesian coordinates, etc. 'Unaltered' means to exclude those obscurations of the individualities of a and b in x which arise (1) from

It appears, then, that what the third postulate essentially demands of the primary group rule is: That in the element which results from combining any two elements, the individuating marks of these two elements shall neither be cancelled by an eliminative comparison with each other nor obscured by intrusion of alien parameters; that a and b shall enter into x only as objects defined in independence of each other by reference to an arbitrary origin of absolute values, which we may represent by the symbol o.

Herein is implied that as members of the group, the elements have no relations to each other which are not themselves elements. Points, related by distances, can not constitute a group; but distances, related by distances, may do so. Now any element k may be the medium of relation for many pairs, $R(n, k, p)$, $R(p, k, r)$, $R(r, k, t)$, etc., but in the pair with itself, $R(i, k, k)$, which is, in a special sense, a declaration of its own identity, or absolute value, it is clear that the absolute value of i is no other than o, which thus appears a member of the assemblage.[11]

It may now readily be shown that the other elements are disposed symmetrically about o, in such wise that for every element a there is an element a', such that $R(a, a', o)$ and also $R(a', a, o)$. The center of symmetry in the assemblage thus corresponds to the center of self-determinations in the group. The element i is a sort of pivot upon which turns the correspondence of the two domains we

---

relativity, and (2) from intrusion of alien variables or alien rules. Thus, $x = (a + b + 2)$, or $x = 2ab$, may be associative; but $x = 2(a + b)$, and $x = ab + 2$ can not be; neither, $x = a + b + b$, nor $x = ab^2$.

I am not yet able to state a necessary and sufficient condition for an associative rule in mathematical form. Such formulation might be attempted on the following lines. Consider $R((a, b), c, v)$ and $R(a, (b, c), u)$. To obtain v, in the first complex triad, a and b have been subjected twice to the group operation—which let us call f—while c has been subjected to it but once: a fact which might be symbolized by $v = ff(a), ff(b), f(c)$. In the second complex triad, $u = f(a), ff(b), ff(c)$. The condition that u shall be always identical to v is that ff shall be neither more nor less than f, in regard to the specifications by which a and b are individuated.

[11] As such member, it has, like k, a triad which in a special sense defines it, $R(o, o, o)$; and enters into an indefinite number of triads as a merely relative 'absolute'—$R(m, o, m)$, $R(n, o, n)$, etc. Thus, with the establishment of the one arbitrary origin, are established as many secondary origins as there are elements; potential origins, if you will, which by their varying departures from the primary origin are analogous to the 'potential' points in a field of force. This transferableness of origin has sometimes been taken as the essential meaning of the associative law, addition, for instance, being explained as follows: in $R(a, b, x)$, b is counted from the transferred origin defined in $R(a, o, a)$. Such explanations are accurate but derivative, inasmuch as the origin o is logically distinct from the element o, for which the symbol i is proper.

are comparing. It is the navel of the group; and the function of the third postulate is to insist upon this connection as the group process approximates absolute freedom and generality, and the group relation approximates perfect independence of the individual,—to insist upon this connection with an arbitrary origin, without which the required *base* for determining the group domain through the group process alone can not be defined.

We describe a group, then, in sum, as the domain of a threefold process, in which two terms of a triad constitute data necessary and sufficient to determine the third; the domain being coincident with the domain of a given dyadic relation defining an assemblage of objects.

The writer is conscious that these formulations are descriptive rather than mathematical. It has been his aim to express the spirit and *Motivirung* of the group concept, within the limits of accuracy, rather than to expound its machinery; and the resulting analysis, which perforce leaves many questions unanswered, may be sufficient to show in what direction we must look for the epistemological service of the group concept.

WILLIAM ERNEST HOCKING.

HARVARD UNIVERSITY.

LINGUISTIC STANDARDS

THE teaching of English in our schools and colleges has concerned itself very little with the theory of linguistic usage, the disadvantages of which omission are apparent in the light of recent psychological advances.

That a historical linguistic standard is necessary is everywhere admitted; use is the law of language. But to the inevitable query 'whose use?' the exponents of the present methods of teaching English give a far from satisfactory reply, not the more so for its practical unanimity. The present historical standard is decidedly conservative in character, exercising the greatest vigilance against all encroachment upon the recognized linguistic material. The author of the usual 'Principles of Rhetoric' resists the technical terminology with the same vigor with which he opposes the dialect of the submerged tenth. But the former arises out of growing needs of the language, while the latter is composed of figurative terms for which there are already proper equivalents. There are many cases in which the condemned technical avoids ambiguities in the sanctioned popular parlance. Under the historical standard in its present form